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Local Artin Root Numbers Associated to Some Classical Polynomials

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INTRODUCTION

Let E be an algebraic number field. The Witt class $\langle E \rangle$ of the quadratic form $\mathrm{Tr}_{E/\mathbb{Q}}(X^2)$ in the Witt ring $W(\mathbb{Q})$ contains important information related to $E/\mathbb{Q}$. The local Artin root numbers of the Galois representation defining the zeta function of E have an explicit connection with the Hasse–Witt invariant of $\langle E \rangle$. Moreover, these local Artin root numbers are related to Weil's additive characters of $\langle E \rangle$ (cf. [2]). The main fact of this relationship is that they can be interpreted as a second Stiefel–Whitney class, as it was proved by Deligne [3] and Serre [10].

Concerning effective determination of these invariants from a defining polynomial of E only few examples of these computations have been carried out [2, Section 10]. Explicit formulas for the local Artin root numbers of abelian Galois representation were given by Tate in [11, Section 1].

Feit in [4] examines trace forms associated to some classical polynomials. He studies the Hasse–Witt invariant associated to some generalized Laguerre polynomials.

In this paper we consider Hilbert's polynomials [5] realizing the alternating group A_n and the exponential Taylor polynomials considered by Schur [8] realizing A_n or S_n. We determine, for each one of these polynomials, the local Artin root numbers of the associated Galois representation. As a consequence Weil's additive characters of the respective Witt classes are computed.

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1. Local Root Numbers, Hasse–Witt Invariants, and Additive Characters

Let $f(X) \in \mathbb{Z}[X]$ be a monic irreducible polynomial. Let

$$E = \mathbb{Q}[X]/f(X)$$

be the simple extension of $\mathbb{Q}$ of degree n defined by $f(X)$.

The transitive permutation representation

$$\rho(E)\colon G_{\mathbb{Q}} \to S_n \subset O(n)$$

gives a real orthogonal representation of $G_{\mathbb{Q}}$. Let $W_p(\mathbb{Q}, \rho(E))$ be the local Artin root numbers associated to the enlarged Artin L-function of $\rho(E)$ (cf. [6, 11]).

Following Deligne [11, Theorem 3], the normalized local root numbers can be interpreted as the local part of the second Stiefel–Whitney class of $\rho(E)$. Serre [10, Theorem 1] gives a formula connecting the Hasse–Witt invariant of the trace form $\mathrm{Tr}_{E/\mathbb{Q}}(X^2)$ with the second Stiefel–Whitney class of $\rho(E)$. From this two results we obtain that

$$\frac{W_p(\mathbb{Q}, \rho(E))}{W_p(\mathbb{Q}, \det \rho(E))} = (2, d_f)_p \cdot h_p(E/\mathbb{Q}),$$

where $h_p(E/\mathbb{Q})$ is the Hasse–Witt symbol at the prime p of the quadratic space $(E, \mathrm{Tr}_{E/\mathbb{Q}}(X^2))$; $W_p(\mathbb{Q}, \det \rho(E))$ is the local Artin root number associated to the one-dimensional representation $\det \rho(E)\colon G_{\mathbb{Q}} \to O(1)$; d_f denotes the discriminant of $f(X)$ (cf. [2, 7]).

On the other hand, let $\langle E \rangle$ be the Witt class of the trace form $\mathrm{Tr}_{E/\mathbb{Q}}(X^2)$ in the Witt ring $W(\mathbb{Q})$. Recently Conner and Yui determine Weil's additive characters γ_p of $\langle E \rangle$ in terms of the local Artin root numbers of $\rho(E)$. These additive characters appear in Weil's formulation of the reciprocity law (for details see [12, p. 179; 2, Sections 5, 6]).

In [2, Theorem 6.2] it is proved that

$$\gamma_p\langle E \rangle = W_p(\mathbb{Q}, \rho(E))^{-1} (\gamma_p\langle 1_{\mathbb{Q}} \rangle)^n,$$

and

$$\gamma_p\langle 1_{\mathbb{Q}} \rangle = \begin{cases} \xi, & \text{if } p = 2 \\ \xi^{-1}, & \text{if } p = \infty \\ 1, & \text{otherwise,} \end{cases}$$

where $\xi = \exp(2\pi i/8)$.

Consequently the computation of the additive characters of $\langle E \rangle$ is equivalent to the computation of the local Artin root numbers of $\rho(E)$.

2. Hilbert's Polynomials for A_n

We recall Hilbert's construction of irreducible polynomials $F(T, X) \in \mathbb{Q}(T)[X]$ having Galois group over $\mathbb{Q}(T)$ isomorphic to the alternating group A_n. If n is even, let $r = (n-2)/2$ $(n > 2)$ and $a_1, ..., a_r$ be positive integers $a_i \neq a_j$. Let $f(X) \in \mathbb{Q}[X]$ be the only polynomial satisfying $f(0) = 0$ and $f'(X) = nX(X-a_1)^2 \cdots (X-a_r)^2$. The polynomial $F(T, X)$ is in this case:

$$F(T, X) = f(X) + (-1)^{n/2} T^2.$$

If n is odd, let $r = (n-1)/2$, $a_1, ..., a_r$ be positive integers $a_i \neq a_j$ and $a = -1/\sum_{k=1}^{r} (a_k)^{-2}$. Let $g(X) = (n-1)(X-a) \prod_{i=1}^{r} (X-a_i)^2$. There exists a polynomial $f(X) \in \mathbb{Q}[X]$ of degree n such that

$$g(X) = Xf'(X) - f(X).$$

The polynomial $F(T, X)$ is in this case,

$$F(T, X) = f(X) + ((-1)^{(n-1)/2} T^2 - f'(a))X.$$

Let $E_T = \mathbb{Q}(T)[X]/F(T, X)$ and denote by $\langle E_T \rangle$ the Witt class of the trace form $\mathrm{Tr}_{E_T/\mathbb{Q}(T)}(X^2)$ in the Witt ring $W(\mathbb{Q}(T))$.

Theorem 2.1. *The Hasse–Witt invariant of* $\langle E_T \rangle$ *is*

$$h(E_T/\mathbb{Q}(T)) = (-1, -1)^{n(n-1)(n-2)(n-3)/8} \cdot (n^{n-1}(n-1)^n, (-1)^{n(n-1)/2}).$$

Let θ be the image of X in E_T. In order to compute the Hasse–Witt invariant of $\langle E_T \rangle$ we need to diagonalize the quadratic form $\mathrm{Tr}_{E_T/\mathbb{Q}(T)}(X^2)$. First, we give some technical results.

Lemma 2.2. *If n is even,*

$$\mathrm{Tr}\left(\theta^i \prod_{k=1}^{r} (\theta - a_k)^{-2}\right) = \begin{cases} 0, & \text{if } 0 \leqslant i \leqslant n-3, \\ n, & \text{if } i = n-2. \end{cases}$$

If n is odd,

$$\mathrm{Tr}\left(\theta^i \prod_{k=1}^{r} (\theta - a_k)^{-2}\right) = \begin{cases} 0, & \text{if } 1 \leqslant i \leqslant n-2, \\ n-1, & \text{if } i = n-1. \end{cases}$$

Proof. Suppose n even, then

$$f'(X) = F'(T, X) = nX(X-a_1)^2 \cdots (X-a_r)^2.$$

By Euler's formula [9, Chap. III, Section 6] we have

$$\mathrm{Tr}(\theta^i/n\theta(\theta-a_1)^2\cdots(\theta-a_r)^2)=\begin{cases}0, & \text{if } 0\leqslant i\leqslant n-2\\ 1, & \text{if } i=n-1.\end{cases}$$

Hence the first assertion is proved.

In the odd case,

$$F'(T,X)=f'(X)+((-1)^{(n-1)/2}\,T^2-f'(a))$$

and

$$f(\theta)=-((-1)^{(n-1)/2}\,T^2-f'(a))\theta.$$

Then

$$\theta F'(T,\theta)=\theta f'(\theta)-f(\theta)=g(\theta)=(n-1)(\theta-a)\prod_{k=1}^{r}(X-a_k)^2.$$

Hence by Euler's formula, we have

$$\begin{aligned}\mathrm{Tr}(\theta^i/F'(T,\theta))&=\mathrm{Tr}\left(\theta^{i+1}/(n-1)(\theta-a)\prod_{k=1}^{r}(\theta-a_k)^2\right)\\&=\begin{cases}0, & \text{if } 0\leqslant i\leqslant n-2\\ 1, & \text{if } i=n-1.\end{cases}\end{aligned}$$

That is,

$$\mathrm{Tr}\left(\theta^i(\theta-a)^{-1}\prod_{k=1}^{r}(\theta-a_k)^{-2}\right)=\begin{cases}0, & \text{if } 1\leqslant i\leqslant n-1\\ n-1, & \text{if } i=n.\end{cases}$$

Therefore,

$$\begin{aligned}&\mathrm{Tr}\left(\theta^i\prod_{k=1}^{r}(\theta-a_k)^{-2}\right)\\&\quad=\mathrm{Tr}\left(\theta^{i+1}(\theta-a)^{-1}\prod_{k=1}^{r}(\theta-a_k)^{-2}\right)\\&\qquad-a\,\mathrm{Tr}\left(\theta^i(\theta-a)^{-1}\prod_{k=1}^{r}(\theta-a_k)^{-2}\right)\\&\quad=\begin{cases}0, & \text{if } 1\leqslant i\leqslant n-2,\\ n-1, & \text{if } i=n-1.\end{cases}\end{aligned}$$

Proof of Theorem 2.1.

Case n even. Consider the linearly independent vectors $e_{2i} = \theta^i(\theta - a_1)^{-1} \cdot \cdots \cdot (\theta - a_r)^{-1}$, $0 \leqslant i \leqslant (n-2)/2$. By Lemma 2.2, e_{2i}, $0 \leqslant i \leqslant (n-2)/2$, are pair-wise orthogonal vectors and isotropic vectors if $0 \leqslant i \leqslant (n-4)/2$. Moreover, $\mathrm{Tr}(e_{n-2}^2) = n$. Therefore, the quadratic space E factorizes as

$$E = \langle e_{n-2} \rangle \perp (n-2)/2H \perp \langle v \rangle,$$

where H is an hyperbolic plane and $\mathrm{Tr}(v^2) = (-1)^{(n-2)/2}\, n$. Therefore,

$$h(E_T/\mathbb{Q}(T)) = (-1, -1)^{n(n-2)/8} \cdot (n, (-1)^{n/2}).$$

Case n odd. Consider the linearly independent vectors $e_{2i} = \theta^i(\theta - a_1)^{-1} \cdot \cdots \cdot (\theta - a_r)^{-1}$, $0 \leqslant i \leqslant (n-1)/2$. By Lemma 2.2, if $0 \leqslant i \leqslant (n-1)/2$, they are pair-wise orthogonal vectors; if $1 \leqslant i \leqslant (n-3)/2$, they are isotropic vectors and $\mathrm{Tr}(e_{n-1}^2) = n-1$. On the other hand,

$$\mathrm{Tr}(e_0^2) = -a \,\mathrm{Tr}\left((\theta - a)^{-1} \prod_{k=1}^{r} (\theta - a_k)^{-2} \right).$$

By Lemma 2.2,

$$0 = \mathrm{Tr}\left(\theta(\theta - a)^{-1} \prod_{k=1}^{r} (\theta - a_k)^{-2} \right) = \mathrm{Tr}\left(f(\theta)(\theta - a)^{-1} \prod_{k=1}^{r} (\theta - a_k)^{-2} \right)$$

$$= a \prod_{k=1}^{r} a_k^2 (n-1) \,\mathrm{Tr}\left((\theta - a)^{-1} \prod_{k=1}^{r} (\theta - a_k)^{-2} \right) + n - 1.$$

Hence $\mathrm{Tr}(e_0^2) = 1 \in \mathbb{Q}(T)^*/\mathbb{Q}(T)^{*2}$. Therefore the quadratic space E factorizes as follows:

$$E = \langle e_0 \rangle \perp \langle e_{n-1} \rangle \perp \langle v \rangle \perp (n-3)/2H,$$

where H is an hyperbolic plane and $\mathrm{Tr}(v^2) = (n-1)(-1)^{(n-3)/2}$. Therefore

$$h(E_T/\mathbb{Q}(T)) = (-1, -1)^{(n-1)(n-3)/8} \cdot (n-1, (-1)^{(n-1)/2}).$$

By Hilbert's irreducibility theorem, there exist infinitely many integral values t of T such that $F_t(X) \in \mathbb{Q}[X]$ is irreducible with Galois group over $\mathbb{Q}$ isomorphic to A_n. Let

$$E = \mathbb{Q}[X]/F_t(X).$$

THEOREM 2.3. *The local Artin root numbers of the real orthogonal representation $\rho(E)$ of $G_{\mathbb{Q}}$ are*

$$W_p(\mathbb{Q}, \rho(E)) = \begin{cases} (-1)^{n(n-1)(n-2)(n-3)/8}, & \text{if } p=\infty \\ (-1)^{n(n-1)(p-1)\, v_p(n^{n-1}(n-1)^n)/4}, & \text{if } p \text{ is an odd finite prime.} \end{cases}$$

Proof. Since the discriminant $d_{F_t(X)} = 1 \in \mathbb{Q}^*/\mathbb{Q}^{*2}$, $W_p(\mathbb{Q}, \det \rho(E)) = 1$. Then $W_p(\mathbb{Q}, \rho(E)) = h_p(E/\mathbb{Q})$. Now we apply Theorem 2.1. If $p = \infty$, $W_\infty(\mathbb{Q}, \rho(E)) = (-1, -1)_\infty^{n(n-1)(n-2)(n-3)/8}$. If p is an odd finite prime

$$W_p(\mathbb{Q}, \rho(E)) = (p, (-1)^{n(n-1)/2})_p^{v_p(n^{n-1}(n-1)^n)}.$$

Remark 2.4. Since the global Artin root number of an orthogonal real representation is trivial [11, p. 124], the local Artin root number of $\rho(E)$ at the prime 2 is

$$W_2(\mathbb{Q}, \rho(E)) = \prod_{p \neq 2} W_p(\mathbb{Q}, \rho(E)).$$

COROLLARY 2.5. *The local Artin root numbers of $\rho(E)$ are trivial for all p if and only if $n \equiv 0$, 1 (mod 8), $n \equiv 2$ (mod 8) and n is a sum of two squares or $n \equiv 3$ (mod 8) and $n-1$ is a sum of two squares.*

Moreover, since the second Stiefel–Whitney class of $\rho(E_T)$ may be interpreted as the obstruction to a Galois embedding problem [10, Section 3], we obtain

COROLLARY 2.6. *Let $\overline{\mathbb{Q}(T)}$ be an algebraic closure of $\mathbb{Q}(T)$. Let $N/\mathbb{Q}(T)$ be the splitting field of Hilbert's polynomial $F(T, X)$. There exist a Galois extension $\tilde{N}$ of $\mathbb{Q}(T)$ such that $\tilde{N} \supset N \supset \mathbb{Q}(T)$ with Galois group $\tilde{A}_n$, the double cover of A_n, if and only if $n \equiv 0$, 1 (mod 8), $n \equiv 2$ (mod 8) and n is a sum of two squares or $n \equiv 3$ (mod 8) and $n-1$ is a sum of two squares.*

3. EXPONENTIAL TAYLOR POLYNOMIALS

Let

$$f(X) = 1 + X + \frac{X^2}{2!} + \cdots + \frac{X^n}{n!}$$

denote the nth-Taylor polynomial of the exponential function. Schur [8] proved that the Galois group of $f(X)$ over $\mathbb{Q}$ is A_n if 4 divides n and the symmetric group S_n otherwise (see [1] for a recent exposition of this result).

Let $E = \mathbb{Q}[X]/f(X)$ and $\langle E \rangle \in W(\mathbb{Q})$ denote the Witt class of the absolute trace form $\mathrm{Tr}_{E/\mathbb{Q}}(X^2)$.

THEOREM 3.1. (i) *The discriminant of* $\langle E\rangle$ *is*

$$\operatorname{dis}\langle E\rangle=\begin{cases}1, & \text{if } n \text{ is even}\\ n!, & \text{if } n \text{ is odd.}\end{cases}$$

(ii) *The Hasse–Witt invariant of* $\langle E\rangle$ *is*

$$h(E/\mathbb{Q})=(-1,-1)^{n(n-1)(n-2)(n-3)/8}\cdot((-1)^{n(n-1)/2},\operatorname{dis}\langle E\rangle).$$

Proof. First we diagonalize the quadratic space E. Let θ be the image of X in E. Clearly, θ^{-i}, $1\leqslant i\leqslant n$, is a basis of E. The irreducible polynomial of θ^{-1} is

$$g(X)=\sum_{i=0}^{n}\frac{1}{i!}X^{n-i}.$$

By Newton's formulae, we have

$$\operatorname{Tr}(1)=n,\qquad \operatorname{Tr}(\theta^{-1})=-1,\qquad \operatorname{Tr}(\theta^{-i})=0,\qquad 2\leqslant i\leqslant n,$$

and $\operatorname{Tr}(\theta^{-(n+1)})=1/n!$.

If n is even, θ^{-i}, $1\leqslant i\leqslant n/2$ are pair-wise orthogonal isotropic vectors. Then E factorizes as

$$E=n/2H,$$

where H is an hyperbolic space. Therefore,

$$\operatorname{dis}\langle E\rangle=(-1)^{n/2}\cdot(-1)^{n/2}=1,$$
$$h(E/\mathbb{Q})=(-1,-1)^{n(n-2)/8}.$$

If n is odd, θ^{-i}, $1\leqslant i\leqslant (n+1)/2$ are pair-wise orthogonal vectors and θ^{-i}, $1\leqslant i\leqslant (n-1)/2$ are isotropic vectors. Then E factorizes as

$$E=(n-1)/2H\perp\langle\theta^{-(n+1)/2}\rangle,$$

where H is an hyperbolic space. Therefore,

$$\operatorname{dis}\langle E\rangle=(-1)^{(n-1)/2}\cdot(-1)^{(n-1)/2}\,n!=n!$$
$$h(E/\mathbb{Q})=(-1,-1)^{(n-1)(n-3)/8}\cdot((-1)^{(n-1)/2},n!).$$

THEOREM 3.2. *The local Artin root numbers of* $\rho(E)$ *are*:

If n *is even*,

$$W_p(\mathbb{Q},\rho(E))=\begin{cases}1, & \text{if } p \text{ is an odd finite prime}\\ (-i)^{n/2}, & \text{if } p=\infty.\end{cases}$$

If n is odd, let m be a square-free integer such that $n! = m \in \mathbb{Q}^*/\mathbb{Q}^{*2}$,

$$W_p(\mathbb{Q}, \rho(E)) = \begin{cases} 1, & \text{if } p \neq 2, \infty;\ p \nmid m \\ \left(-\dfrac{2m}{p}, p\right)_p, & \text{if } p \mid m \text{ and } p \equiv 1 \pmod 4 \\ \left(-\dfrac{2m}{p}, p\right)_p i, & \text{if } p \mid m \text{ and } p \equiv 3 \pmod 4 \\ (-i)^{(n-1)/2}, & \text{if } p = \infty. \end{cases}$$

Proof. Let $\alpha \in \mathbb{Q}^*/\mathbb{Q}^{*2}$, $\rho(\alpha)$ denotes the 1-orthogonal representation

$$\rho(\alpha): G_{\mathbb{Q}} \to O(1)$$
$$\rho(\alpha)(\sigma) = \sigma(\sqrt{\alpha})/\sqrt{\alpha}, \qquad \text{for all} \quad \sigma \in G_{\mathbb{Q}}.$$

Clearly

$$W_p(\mathbb{Q}, \rho(\alpha)) = W(\mathbb{Q}_p, \chi(\alpha)),$$

where χ is the real quadratic character on $\mathbb{Q}_p$ given by the Hilbert symbol,

$$\chi(\alpha): \mathbb{Q}_p^* \to \mathbb{Z}^*, \qquad x \mapsto (x, \alpha)_p.$$

Since $W_p(\mathbb{Q}, \det \rho(E)) = W_p(\mathbb{Q}, \rho(d_f))$, we have

$$W_p(\mathbb{Q}, \rho(E)) = (2, d_f)_p\, h_p(E/\mathbb{Q})\, W_p(\mathbb{Q}, \rho(d_f)).$$

The values of the local root numbers for the quadratic character $\chi(d_f)$ can be determined from [11, p. 94].

If n is even, $d_f = (-1)^{n/2}$. Then

$$\begin{aligned} W_p(\mathbb{Q}, \rho(E)) &= (2, (-1)^{n/2})_p \cdot (-1, -1)_p^{n(n-2)/8} \cdot W_p(\mathbb{Q}, \rho((-1)^{n/2})) \\ &= \begin{cases} 1, & \text{if } p \text{ is an odd finite prime,} \\ (-i)^{n/2}, & \text{if } p = \infty. \end{cases} \end{aligned}$$

If n is odd, $d_f = (-1)^{(n-1)/2}\, n!$. Then

$$\begin{aligned} W_p(\mathbb{Q}, \rho(E)) = (2(-1)^{(n-1)/2}, n!)_p \cdot (-1, -1)_p^{(n-1)(n-3)/8} \\ \cdot W_p(\mathbb{Q}, \rho((-1)^{(n-1)/2}\, n!). \end{aligned}$$

The values of $W_p(\mathbb{Q}, \rho((-1)^{(n-1)/2} n!)$ are given by

$$W_p(\mathbb{Q}, \rho((-1)^{(n-1)/2} n!))$$

$$= \begin{cases} 1, & \text{if } p=\infty, n \equiv 1 \pmod 4 \\ -i, & \text{if } p=\infty, n \equiv 3 \pmod 4 \\ ((-1)^{(n-3)/2} m/p, p)_p, & \text{if } p \mid m, p \equiv 1 \pmod 4 \\ ((-1)^{(n-3)/2} m/p, p)_p\, i, & \text{if } p \mid m, p \equiv 3 \pmod 4 \\ 1, & \text{if } p \neq 2, \infty; p \nmid m. \end{cases}$$

(cf. [2, Section 4] for a detailed description).

Remark 3.3. Since the global Artin root number is trivial in this case [11, p. 124], the value of the local root number at the prime $p=2$ is

$$W_2(\mathbb{Q}, \rho(E)) = \prod_{p \neq 2} W_p(\mathbb{Q}, \rho(E)).$$

COROLLARY 3.4. *Let $\bar{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$. Let N be the splitting field of the exponential Taylor polynomial $f(X)$ and G its Galois group over $\mathbb{Q}$. There exists a Galois extension $\tilde{N}$ of $\mathbb{Q}$ such that $\tilde{N} \supset N \supset \mathbb{Q}$ with Galois group a stem cover of G if and only if $n \equiv 0$, 2, or 6 (mod 8).*

Proof. If $4 \mid n$, $G = A_n$ and there exists only one stem cover $\tilde{A}_n$ of A_n. Otherwise $G = S_n$ and there exist two stem covers 2^+S_n, 2^-S_n of S_n. Let $e\colon G_{\mathbb{Q}} \to G(N/\mathbb{Q}) = G$. The obstructions to the associated embedding problems as elements of $H^2(G_{\mathbb{Q}}, \mathbb{Z}/2\mathbb{Z})$ are [10, Section 3],

$$e^*(a_n) = h(E/\mathbb{Q}),$$

$$e^*(s_n^+) = h(E/\mathbb{Q}) \cdot (2, d_f),$$

$$e^*(s_n^-) = h(E/\mathbb{Q}) \cdot (-2, d_f).$$

By applying Theorem 3.1, the result follows.

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